JACOBIANS

If u and v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u, v and denoted by $J(u,v) = \frac{\partial(u,v)}{\partial(x,y)}$

Similarly

$$J(u\,v\,w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties of Jacobians

(1) Chain rule for Jacobians

If $u(r,s)$ and $v(r,s)$ i.e u and v are fun. of r, s and r, s are fun. of (x,y) then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

(2) If J_1 is Jacobian of (u,v) w.r.t (x,y) and J_2 is the Jacobian of (x,y) w.r.t (u,v) then $J_1 J_2 = 1$ i.e

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1.$$

Prob. 1. If $u = x(1-y)$, $v = xy$ Prove that $JJ' = 1$.

2. If $x = r\cos\theta$, $y = r\sin\theta$ Show that $\frac{\partial(x,y)}{\partial(r,\theta)} = r$ and $JJ' = 1$

3. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$, Show that

$$\frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)} = 4.$$