B.Tech.

Third Semester Examination, 2010-2011

Discrete Structure (CSE-203-F)

Note : Attempt **five** questions, Q. No. 1 is compulsory, and one question from each of the four Sections. All questions carry equal marks.

Q. 1. (a) Define multiset and power set.

Ans. Multiset : Multisets are sets where an element can occur as a member more than once. For example :

$$A = \{a, a, a, b, b, c\}$$

$$B = \{a, a, a, a, b, b, b, d, d\}$$

are multisets. The multisets A and B can also be written as :

$$A = \{3.a, 2.b, 1.c\} \text{ and}$$

$$B = \{4.a, 3.b, 2.d\}$$

Power Set : If S is any set, then the family of all the subsets of S is called the power set of S. The power set of S is denoted by $P(S)$. Symbolically

$$P(S) = \{T : T \in S\}$$

Thus, ϕ and S are both element of $P(S)$. If set S is finite and contain n elements, then the power set of S will contain 2^n elements.

e.g., if $A = \{1, 2\}$, then $P(A) = \{\{1\}, \{2\}, \{1, 2\}, \{\ \}\}$.

Q. 1. (b) Define Cartesian product of sets and equivalence relation.

Ans. Cartesian Product of Sets : Let A and B be sets. Cartesian product of A and B, denoted by $A \times B$, is defined as :

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Thus, $A \times B$ is the set of all possible ordered pairs whose first component comes from A and whose second component comes from B.

Equilvalence Relation : A relation on a set A is called equivalence relation if it is reflexive, symmetric and transitive, i.e., R is an equivalence relation on A if it has the following three properties :

(i) $(a, a) \in R$ for all $a \in A$ (reflexive)

(ii) $(a, b) \in R$ implies $(b, a) \in R$ (symmetric)

(iii) (a, b) and $(b, c) \in R$ imply $(a, c) \in R$ (transitive).

Q. 1. (c) If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x-1)$.

Ans. $f(x) = 3x^4 - 5x^2 + 9$

Then $f(x-1) = 3(x-1)^4 - 5(x-1)^2 + 9$

$$= 3(x^4 - 4x^3 + 6x^2 - 4x + 1) + 5(x^2 - 2x + 1) + 9$$

$$= 3x^4 - 12x^3 + 23x^2 - 22x + 17$$

Q. 1. (d) Define partial order relation and lattice.

Ans. Partial Order Relation : A relation R on a set S is called a partial ordering if it is reflexive, antisymmetric and transitive. That is :

(i) aRa for all $a \in S$ (reflexivity)

(ii) aRb and $bRa \Rightarrow a = b$ (antisymmetry)

(iii) aRb and $bRc \Rightarrow aRc$ (transitivity)

Lattice : A poset $(P \leq)$ is called a lattice if every 2-element subset of P has both a least upper bound, a greatest lower bound, i..e., if lub (x, y) and glb (x, y) exist for every x and y in P. This is denoted as :

$$x \vee y = \text{lub}\{x, y\} \qquad (x \text{ join } y)$$

$$x \wedge y = \text{glb}\{x, y\} \qquad (x \text{ meet } y)$$

Every chain is a lattice. Since any two elements a, b of a chain are comparable. We find

$$x \vee y = \text{lub}(x, y) = y$$

$$x \wedge y = \text{glb}(x, y) = x$$

Q. (e) Define AP and GP. Also write the formula for sum of *n* terms in AP and GP.

Ans. AP (Arithmetic Progression) : An arithmetic progression (AP) or arithmetic sequence is a sequence of numbers such that the difference of any two successive members of the sequence is a constant. For instance, the sequence 3, 5, 7, 9, 11, 13, ... is an arithmetic progression with common difference 2.

If the initial term of an arithmetic progression is a_1 and the common difference of successive members is d, then the n^{th} term of the sequence is given by :

$$\boxed{a_n = a_1 + (n-1)\, d}$$

and in general :

$$\boxed{a_n = a_m + (n-m)\, d}$$

GP (Geometric Progression) : A geometric progression (GP) also known as a geometric sequence, is a sequence of numbers where each term after the first is found by multiplying the previous one by a fixed non-zero number called the common ratio. For example, the sequence 2, 6, 18, 54, ... is a GP with common ratio 3.

Thus, in general a GP is :

$$a, ar, ar^2, ar^3, ar^4, \ldots\ldots$$

The n^{th} term of a geometric sequence with initial value 'a' and common ratio 'r' is given by :

$$\boxed{a_n = ar^{n-1}}$$

Q. 1. (f) How many variable names of 8 letters can be formed from the letters a, b, c, d, e, f, g, h, i if no letter is repeated.

Ans. If repetition is not allowed the no. of variable names of 8 letters formed from the letters : $a, b, c, d, e, f, g, h, i$ are :

$$P(n, r) = \frac{n!}{(n-r)!}$$

Given, $n = 9$ and $r = 8$

$\therefore$ $$P(9, 8) = \frac{9!}{(9-8)!} = 362880$$

Q. 1. (g) How many 4-digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed.

Ans. When repetition of digits is allowed the no. of 4-digits numbers that can be formed using the digits 2, 4, 6, 8 are : n^r.

Given, $n = 4$ and $r = 4$

$\therefore$ No. of 4-digit numbers $= n^r$

$= 4^4$

$= 256$

Q. 1. (h) Define Monoid and Semigroup with suitable example.

Ans. Monoid : An algebraic structure $(S, *)$ is called a monoid if the following conditions are satisfied :

(i) The binary operation * is a closed operation. (Closure law)

(ii) The binary operation * is an associative operation. (Associative law)

(iii) There exists an identity element, i.e., for some $e \in S, e * a = a * e = a$ for all $a \in S$.

Thus, a monoid is a semigroup $(S, *)$ that has an identity element.

e.g., If Z be a set of all integers, then the structure $(Z, +)$ is a monoid with identity element 0 and $(Z, \bullet)$ is a monoid with 1 as the identity element.

Semigroup : An algebraic structure $(S, *)$ is called a semigroup if the following conditions are satisfied :

(i) The binary operation * is a closed operation, i.e., $a * b \in S$ for all $a, b \in S$. (Closure law)

(ii) The binary operation * is an associative operation, i.e., $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$. (Associative-law).

e.g., If Z be a set of all integers, then $(Z, +)$ and $(Z, \bullet)$ are semigroup as these two operations are closed associative in Z.

Q. 1. (i) Define Isomorphic graph and Homeomorphic graph.

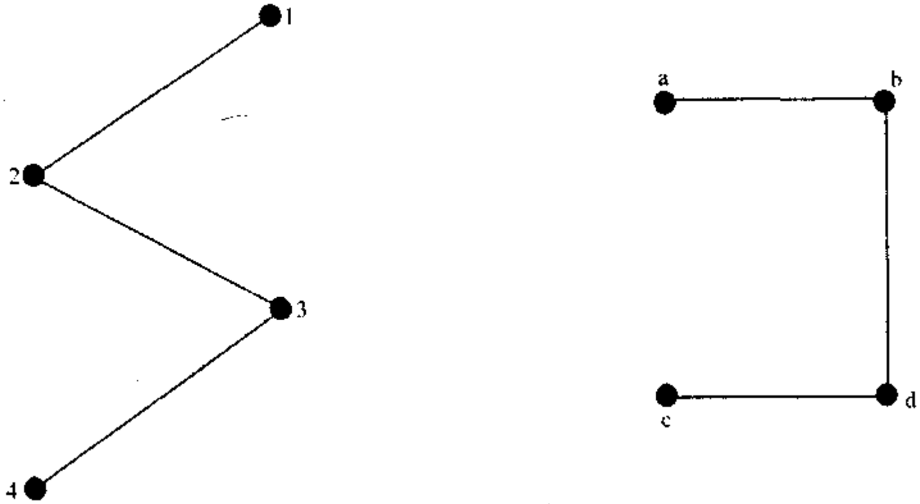
Ans. Isomorphic Graph : Let $G_1 = (v_1, E_1)$ and $G_2 = (v_2, E_2)$ be two graphs. A function $f = v_1 \rightarrow v_2$ is called graph isomorphism if :

(a) f is one-to-one onto, and

(b) for all $a, b \in V_1$ $\{a, b\} \in E$, if and only it $\{f(a), f(b)\} \in E_2$

When such a function exists, G_1 and G_2 are called isomorphic graphs and is written as $G_1 \equiv G_2$.

e.g., The following two graphs are isomorphic :



Here, $V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$, $E(G_1) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$

& $E(G_2) = \{\{a, b\}, \{b, d\}, \{d, c\}\}$

Let function $f : V(G_1) \to V(G_2)$ as

$$f(1) = a, \quad f(2) = b, \quad f(3) = d \quad \text{and} \quad f(4) = c$$

f is clearly one-one and onto, hence an isomorphism.

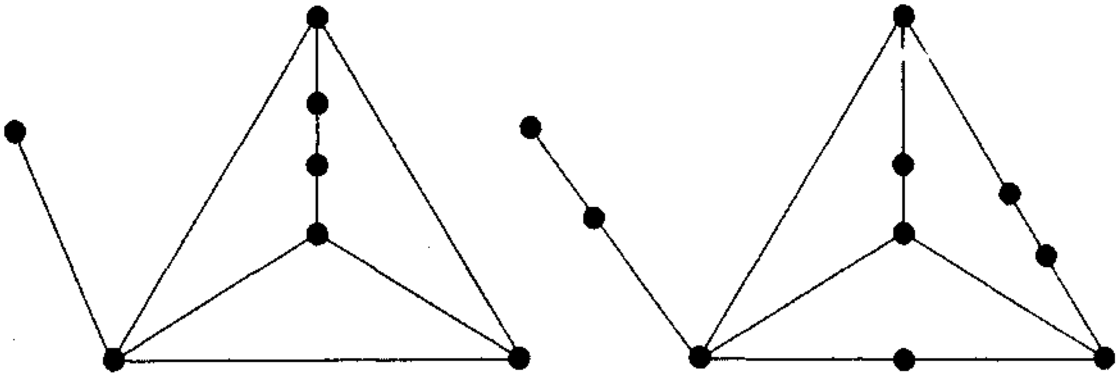
Further, $\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

Hence, G_1 and G_2 are isomorphic.

Homeomorphic Graph : Two graphs are said to be homeomorphic if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges or by the merger of edges in series. Such an operation is called an elementary subdivision. For example, any two cycle graphs are homeomorphic, as are the graphs of fig. :



Section—A

Q. 2. (a) Prove :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Ans.

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let x be any arbitrary element of the set $A \cap (B \cup C)$. Then,

$$
\begin{aligned}
x \in A \cap (B \cup C) &\Rightarrow x \in A \text{ and } x \in (B \cup C) \\
&\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \\
&\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
&\Rightarrow (x \in A \cap B) \text{ or } (x \in A \cap C) \\
&\Rightarrow x \in (A \cap B) \cup (A \cap C)
\end{aligned}
$$

Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ ...(i)

Conversely, let x be any arbitrary element of the set $(A \cap B) \cup (A \cap C)$. Then,

$$
\begin{aligned}
x \in (A \cap B) \cup (A \cap C) &\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \\
&\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\
&\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \\
&\Rightarrow x \in A \cap (B \cup C)
\end{aligned}
$$

or $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ ...(ii)

From equations (i) and (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let x be any arbitrary element of the set $A \cup (B \cap C)$. Then,

$$
\begin{aligned}
x \in A \cup (B \cap C) &\Rightarrow x \in A \text{ or } x \in (B \cap C) \\
&\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\
&\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
&\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \\
&\Rightarrow x \in (A \cup B) \cap (A \cup C)
\end{aligned}
$$

Thus, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ ...(i)

Conversely, let x be any arbitrary element of the set $(A \cup B) \cap (A \cup C)$. Then,

$$
\begin{aligned}
x \in (A \cup B) \cap (A \cup C) &\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \\
&\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
&\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\
&\Rightarrow x \in A \cup (x \in (B \cap C)) \\
&\Rightarrow x \in A \cup (B \cap C)
\end{aligned}
$$

or $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ ...(ii)

From equations (i) and (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Q. 2. (b) Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$

Show that R is an equivalence relation.

Ans. Let $A = \{1, 2, 3, 4\}$ and

$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$

Let $a \in A$, then R is reflexive since $(a, a) \in R \; \forall \; a \in R$

R is reflexive since, $(1, 1), (2, 2), (3, 3), (4, 4) \in R$.

R is symmetric because whenever, $(a, b) \in R$

$\Rightarrow \quad (b, a) \in R$

Here, $\quad (1, 3) \in R$ so is $(3, 1)$

$(2, 4) \in R$ so is $4, 2)$

Therefore, R is a symmetric relation.

R is transitive if and only if (a, b) and $(b, c) \in R$ imply $(a, c) \in R$

Here, $\quad (1, 3)$ and $(3, 1) \in R \Rightarrow (1, 1) \in R$

& $\quad (2, 4)$ and $(4, 2) \in R \Rightarrow (2 \;\; 2) \in R$ and so on.

Thus, R is a transitive relation.

Since, R is reflexive, symmetric and transitive, it is an equivalence relation.

Q. 2. (c) Let $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$.

Find :

(i) *fof* **(ii) *fog*** **(iii) *gof***

Ans. (i)

$$fof = f[f(x)] = f(x^2 + 3x + 1)$$
$$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$$
$$= x^4 + 6x^3 + 11x^2 + 6x + 1 + 3x^2 + 9x + 3 + 1$$
$$= x^4 + 6x^3 + 14x^2 + 15x + 5$$

(ii)

$$fog = f[g(x)] = f(2x - 3)$$
$$= (2x - 3)^2 + 3(2x - 3) + 1$$
$$= 4x^2 - 12x + 9 + 6x - 9 + 1$$
$$= 4x^2 - 3x + 1$$

(iii)

$$gof = g[f(x)] = g|x^2 + 3x - 1|$$
$$= 2(x^2 + 3x + 1) - 3 = \mathbf{2x^2 + 6x - 1}$$

Q. 3. (a) Prove that the statement :

$(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a tautology.

Ans. The truth table of the given proposition is shown below :

p	q	$\sim p$	$\sim q$	$(p \to q)$	$(\sim q \to \sim p)$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since, the truth value is TRUE for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is a tautology.

Q. 3. (b) Prove that $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$.

Ans. The truth table given below shows that these two expressions are logically equivalent, the two columns corresponding to the given two expressions have identical truth values.

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Q. 3. (c) Let P and Q be the relations on set

$A = \{1, 2, 3, 4\}$ defined by

$P = \{(1, 2), (2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (4, 3)\}$ and

$Q = \{(2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)\}$

Find :

(i) POP **(ii) POQ** **(iii) POPOQ**

Ans. Given : $A = \{1, 2, 3, 4\}$ and

$P = \{(1, 2), (2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (4, 3)\}$

$Q = \{(2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)\}$

(i) POP = {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

(ii) POQ = {(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (2, 1), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)}

(iii) POPOQ = PO(POQ)

$\Rightarrow \quad P = \{(1, 2), (2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (4, 3)\}$

$(POQ) = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (2, 1), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$

$\therefore \quad PO(POQ) = \{(1, 2), (1, 3), (1, 4), (1, 1), (2, 2), (2, 3), (2, 4), (2, 1), (3, 2), (3, 3), (3, 4), (3, 1), (4, 2), (4, 3), (4, 4) (4, 1)\}$

Q. 3. (d) Prove that if L be a lattice then $a \wedge b = a$ if and only if $a \vee b = b$.

Ans. Given that if L be a lattice then $a \wedge b = a$ if and only if $a \vee b =.b$.

If L be a lattice then for every a and b in L.

To prove this property, we first have to prove the following two properties :

(i) $a \vee b = b$ if and only if $a \leq b$ and

(ii) $a \wedge b = a$ if and only if $a \leq b$.

(i) Let $a \vee b = b$. Since, $a \leq a \vee b = b$, we get $a \leq b$. Conversely if $a \leq b$, then, since $b \leq b$, b is an upper bound of a and b. So, by definition of lub; we have $a \vee b \leq b$. Again since $a \vee b$ is an upper bound, $b \leq a \vee b$, so $a \vee b = b$. Hence proved.

(ii) Similarly, $a \wedge b = a$

By combining equations (i) and (ii) we get :

$$a \wedge b = a \quad \text{iff} \quad a \vee b = b$$

Section—B

Q. 4. (a) Determine the number of permutations that can be made out of the letters of the word 'PROGRAMMING'.

Ans. The word 'PROGRAMMING' consists of 11 letters which can be arranged in :

Total no. of letters in the word 'PROGRAMMING' is 11, out of which R occurs twice, G occurs twice, M occurs twice and the rest are all different. Since some of the letters are repeated we need to apply multimonial theorem. Hence the no. of arrangements are :

$$= \frac{11!}{2!2!2!} = 4989600$$

Q. 4. (b) How many 2-digits even number can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits is allowed ?

Ans. To form 2-digits even numbers using digits : 1, 3, 4, 6, 8 when repetition of digits is allowed, the no. must end with an even digit, the even digits here are – 4, 6 and 8.

Therefore, by product rule :

The first digit can be chosen from the five digits : 1, 3, 4, 6 and 8.

The second digit can be any one of the three digits : 4, 6 and 8.

Hence, the no. of 2-digits even no. are $5 \bullet 3 = 15$.

Q. 4. (c) How many ways can we select a software development group of 1 project leader, 5 programmers and 6 data entry operators from a group of 5 project leaders, 20 programmers and 25 data entry operator's ?

Ans. No. of ways to select 1 project leader from a group of 5 project leaders

$$= C(5, 1) = \frac{5!}{1!(5-1)!} = \frac{5 \times 4!}{4!} = 5$$

No. of ways to select 5 programmers from a group of 20 programmers

$$= C(20 5) = \frac{20!}{5!(20-5!)} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15!}{5 \times 4 \times 3 \times 2 \times 15!}$$

$$= 15504$$

No. of ways to select 6 data entry operators from a group of 25 data entry operators

$$C(26, 6) = \frac{25!}{6!(25-6)!}$$

$$= \frac{25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19!}{6 \times 5 \times 4 \times 3 \times 2 \times 19!}$$

$$= 177100$$

∴ Total no. of ways of select a group of 1 project leader, 5 programmer and 6 data entry operators are :

$$= C(5, 1) \times C(20, 5) \times C(25, 6)$$

$$= 5 \times 15504 \times 177100$$

Q. 5. (a) Solve the recurrence relation :

$$a_{r+2} - 3a_{r+1} + 2a_r = 0$$

by the method of generating functions with the initial conditions $a_0 = 2, a_1 = 3$.

Ans. Recurrence relation is

$$a_{r+2} - 3a_{r+1} + 2a_r = 0 \text{ with } a_0 = 2, a_1 = 3$$

The characteristic equations of the recurrence relation is

$$\alpha^2 - 3\alpha + 2 = 0 \Rightarrow (\alpha - 1)(\alpha - 2) = 0 \Rightarrow \alpha = 1, 2$$

The general homogeneous solution of the recurrence relation

$$\alpha^*_{nh} = A_1 (1)^n + A_2 (2)^n \quad \ldots(i)$$

Where A_1 & A_2 are constants.

Putting $n = 0, 1$ in equation (i)

$$a_0 = A_1 + A_2 \quad \Rightarrow \quad A_1 + A_2 = 2 \quad \Rightarrow \quad A_1 = 1, \quad A_2 = 1$$

$$a_1 = A_1 + 2A_2 \qquad A_1 + 2A_2 = 3$$

Hence solution is

$$\alpha^*_{np} = 1 + (2)^n$$

Q. 5. (b) Solve the recurrence relation

$$2a_r - 5a_{r-1} + 2a_{r-2} = 0 \text{ and find}$$

particular solution such that $a_0 = 0, a_1 = 1$.

Ans. The given recurrence relation is

$$2a_r - 5a_{r-1} + 2a_{r-2} = 0 \qquad \& \qquad a_0 = 0,\ a_1 = 1$$

The given characteristic equation of given recurrence relation is

$$2\alpha^2 - 5\alpha + 2 = 0$$

$$\Rightarrow \quad (\alpha - 2)(2\alpha - 1) = 0 \qquad \Rightarrow \qquad \alpha = 2, \frac{1}{2}$$

The general homogeneous solution of recurrence relation is

$$\alpha^*_{nh} = A_1 2^n + A_2 \left(\frac{1}{2}\right)^n$$

Putting $n = 0, 1$

$$a_0 = A_1 + A_2 \qquad \Rightarrow \qquad A_1 + A_2 = 0 \qquad \Rightarrow \qquad A_1 = \frac{2}{3}, \qquad A_2 = -\frac{2}{3}$$

$$a_1 = 2A_1 + \frac{1}{2}A_2 \quad \Rightarrow \quad 2A_1 + \frac{1}{2}A_2 = 1$$

Hence solution is

$$\alpha^*_{nh} = \frac{2}{3}(2)^n - \frac{2}{3}\left(\frac{1}{2}\right)^n$$

Section—C

Q. 6. Define the following with suitable example :

(i) Rings

(ii) Field

(iii) Integral domain

(iv) Normal subgroup

(v) Homomorphism

Ans. (i) Rings : A ring $(R, +, \cdot)$ is a set R together with two binary operations + (addition) and $\cdot$ (multiplication) defined on R such that the following axioms are satisfied :

(R_1) $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$

(R_2) $a + b = b + a$ for all $a, b \in R$

(R_3) There exists an element 0 in R such that $a + 0 = a$ for all $a \in R$.

(R_4) For all $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0$

(R_5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.

(R_6) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$

(R_7) $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$.

We call 0, the zero element of the ring $(R, +, \cdot)$

e.g.,

(i) The set Z of integers under ordinary addition and multiplication is a commutative ring with unity 1. The unit elements of Z are 1 and -1.

(ii) The set $Zn = \{0, 1, 2, \ldots\ldots, n-1\}$ under addition and multiplication modulo n is a commutative ring with unity 1.

(ii) Field : A ring containing at least two elements is called a field if it :

(a) is commutative

(b) has unity and

(c) is such that every non-zero element has multiplicative inverse in R.

That is – a system $(R, +, \cdot)$ is a field if

(i) $(R, +)$ is an abelian group.

(ii) $(R', \cdot)$ is a commutative group, where $R' = R - \{0\}$.

(iii) The distributive laws :

$$a(b+c) = ab + ac$$

$$(b+c)\,a = ba + ca \qquad \text{hold for all } a, b, c \in R.$$

e.g.,

(i) The ring of rational numbers $(Q, +, *)$ is a field since it is a commutative ring with unity and each non-zero element has multiplicative inverse.

(ii) The set R of all real numbers is a field.

(iii) Integral Domain : A ring containing at least two elements is called an Integral domain if it :

(a) is commutative

(b) has unit element and

(c) is without zero divisors.

Thus, in an integral domain a product is 0 only when one of the factors is 0; i.e., $ab = 0$ only when $a = 0$ or $b = 0$.

e.g.,

(i) The ring of integers $(Z, +, \cdot)$ is an integral domain since it is commutative ring with unity and for any two integers a, b, $ab = 0 \Rightarrow a = 0$ or $b = 0$.

(iv) Normal Subgroup : A subgroup H of a group G is said to be normal subgroup of G if $Ha = aH$ for all $a \in G$. Clearly every subgroup of an Abelian group is a normal subgroup.

Thus, a subgroup H of a group G can be defined to be a normal subgroup if

$$g^{-1}hg \in H \;\forall\; h \in H, g \in G$$

e.g., If H is a subgroup of G such that $x^2 \in H$ for every $x \in G$, then H is a normal subgroup of G.

(v) Homomorphism : Let $(G, *)$ and $(G_1, *_1)$ be two groups and f is a function from G into G_1. Then f is called a homomorphism of G into G_1 if for all $a, b \in G$.

$$f(a * b) = f(a) *_1 f(b)$$

e.g.,

Let $G = Z$ and $G' = \{1, -1\}$ the multiplicative group. The mapping $f : G \to G'$ defined by $f(n) = 1$ if n is even and $f(n) = -1$ if n is odd is a group homomorphism, as $f(m+n) = f(m)\,f(n)$ for all $m, n \in Z$.

Q. 7. (a) State and prove Lagrange's theorem.

Ans. Lagrange's Theorem : The order of each sub-group of a finite group G is a divisor of the order of the group G.

Proof : Let H be any sub-group of order m of a finite group G of order n. We consider the left coset decomposition of G relative to H.

We first show that each coset a H consists of m different elements.

Let $H = \{h_1, h_2, \ldots, h_m\}$

Then $ah_1, ah_a, \ldots, ah_m$ are the m members of aH, all distinct.

For, we have

$$ah_i = ah_j \Rightarrow h_i = h_j, \text{ by cancellation law in } G.$$

Since, G is a finite group, the number of distinct left cosets will also be finite, say k. Hence the total no. of elements of all cosets is km which is equal to the total no. of elements of G. Hence,

$$n = mk$$

This shows that m, the order of H, is a divisor of n, the order of the group G.

Q. 7. (b) Define the following with suitable example :

(i) Automorphism

(ii) Cyclic group

(iii) Cosets

Ans. (i) Automorphism : A homomorphism f of a group G into a group G_1 is called a isomorphism of G onto G_1 if f is one-one onto G_1. G and G_1 are said to be isomorphic and denoted by $G \cong G_1$. An isomorphism of a group G onto G is called an automorphism.

(ii) Cyclic Group : A group G is called a cyclic group if, for some $a \in G$, every element of G is of the form a^n, where n is some integer. The element a is then called a generator of G.

If G is a cyclic group generated by a, it is denoted by $G = \langle a \rangle$. The elements of G are in the form :

$$\ldots\ldots a^{-2}, a^{-1}, a^0, 0, a, a^2, a^3 \ldots\ldots$$

There may be more than one generator of a cyclic group.

e.g., The set of integers with respect to + i.e., $(Z, +)$ is a cyclic group, a generator being 1. i.e., each element of G can be expressed as some integral power of 1.

(iii) Cosets : Let H be a subgroup of a group G and let $a \in G$. Then the set $\{a * h : h \in H\}$ is called the left coset generated by a and H and is denoted by aH.

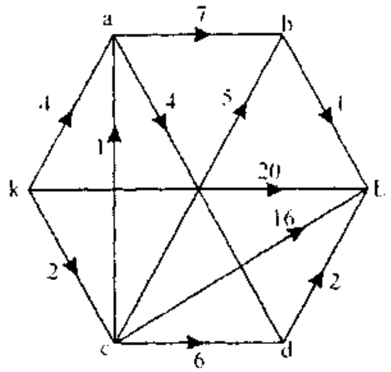
Similarly, the set $Ha = \{h * a : h \in H\}$ is called the right coset and is denoted by Ha. The element a is called a representative of aH and Ha. If the group operation be addition, then the right coset of H in G generated by a is defined as

$$H + a = \{h + a : h \in H\}$$

Similarly, left coset $a + H = \{a + h : h \in H\}$.

Section—D

Q. 8. (a) Find the shortest path from K to L.



Ans. Dijkshtra Algorithm to find the shortest path from K to L :

The initial labelling is given by :

Vertex V	k	b	c	d	a	L
$L(v)$	0	∞	∞	∞	∞	∞
T	$\{k$	b	c	d	a	$L\}$

Iteration 1 : $u = k$ has $L(u) = 0$. T becomes $T - \{k\}$. There are three edges incident with k : a, c and L where $a, c\ L \in T$.

$$L(a) = \min(\text{old } L(a), L(k) + w(ka))$$
$$= \min(\infty, 0+4) = 4$$
$$L(c) = \min(\text{old } L(c), L(k) + w(kc))$$
$$= \min(\infty, 0+2) = 2$$
$$L(L) = \min(\text{old } L(L), L(k) + w(kL))$$
$$= \min(\infty, 0+20) = 20$$

Hence, min. label is $L(c) = 2$.

↓

Vertex	k	b	c	d	a	L
$L(v)$	0	∞	2	∞	4	20
T	$\{b$	c	d	a	$L\}$	

Iteration 2 : $u = c$, the permanent label of c is 2. T becomes $T - \{c\}$. There are four edges incident with c. a, b, L, d where $a, b, L, d \in T$.

$L(a) = \min(4, L(c) + w(ca))$

$= \min(4, 2 + 1) = 3$

$L(b) = \min(\infty, 2 + 5) = 7$

$L(L) = \min(20, 2 + 16) = 18$

$L(d) = \min(\infty, 2 + 6) = 8$

Hence, min. label is $L(a) = 3$.

↓

Vertex	k	b	c	d	a	L
$L(v)$	0	7	2	8	3	18
T	{	b,		d,	a,	L}

Iteration 3 : $u = a$, the permanent label of a is 3. T becomes $T - \{a\}$. There are two edges incident with a: b, d where $b, d \in T$.

$L(b) = \min(7, 3 + 7) = 7$

$L(d) = \min(8, 3 + 4) = 7$

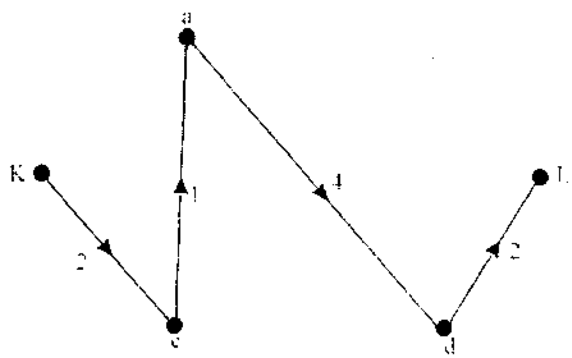
Hence, min. label is $L(d) = 7$.

Vertex	k	b	c	d	a	L
$L(v)$	0	7	2	7	3	18
T	{	b,		d,	,	L}

Iteration 4 : $u = d$, the permanent label of d is 7. T becomes $T - \{d\}$. There is only one edge incident with d ; L where $L \in T$.

$L(L) = \min(18, 7 + 2) = 9$

Now iteration stops. Thus, the shortest distance between k to L is 9 and shortest path is (k, c, a, d, L).

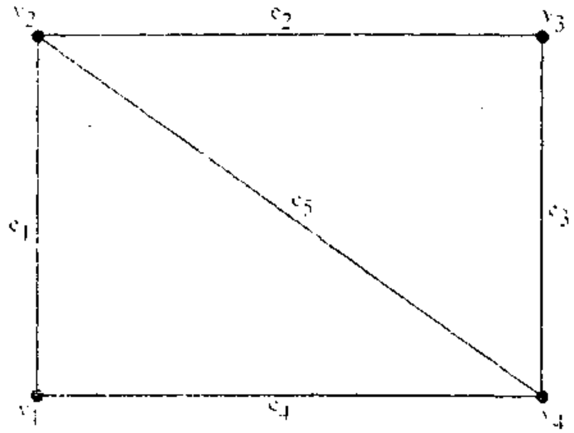


Q. (b) (i) Define Hamilton path and circuit with example.

Ans. Hamilton Path : A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

Hamilton Circuit : A circuit in a graph, G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian circuit.

e.g., The graph shown in fig. has Hamiltonian circuit given by $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$.



It has Hamiltonian path given by $v_1 - v_2 - v_3 - v_4$.

Q. 8. (b) (ii) Define sub-graph.

Ans. Sub-graph : A sub-graph is a graph obtained by removing some of the vertices of a graph G as well as all edges incident with a vertex that was removed, are also removed from the graph.

If G and H are two graphs with vertex sets $V(H)$, $V(G)$ and edge sets $E(H)$ and $E(G)$ respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call H as a subgraph of G.

Q. 8. (b) (iii) What properties should graph possess to qualify as tree ?

Ans. A graph G with n vertices is called a tree if :

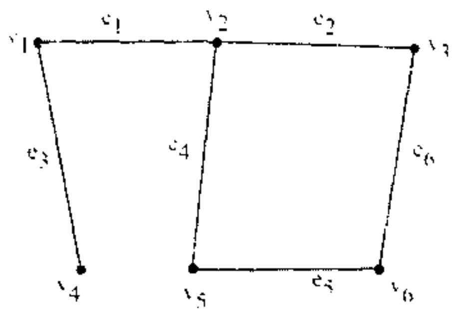
(i) G is connected and has no cycles (acyclic).

(ii) G is connected and has $n - 1$ edges.

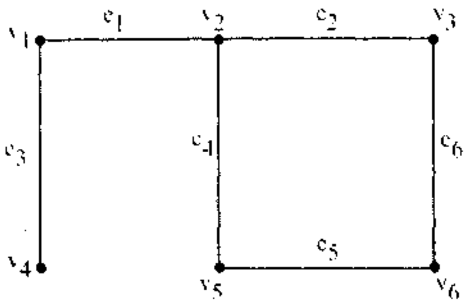
(iii) G is acyclic and has $n - 1$ edges.

(iv) There is exactly one path between every pair of vertices in G.

Q. 9. (a) For the following graph find all the bridges.



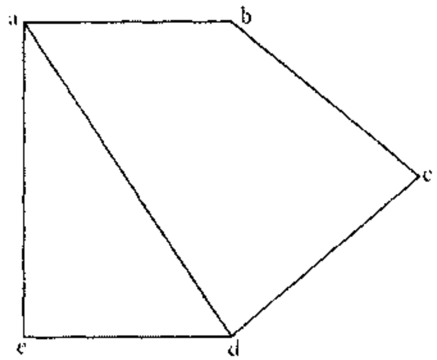
Ans. Given graph is :

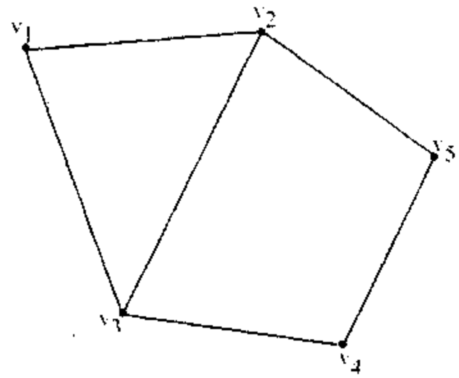


An edge whose removal produces a graph with more connected components than the original graph is a cut edge or bridge. The given fig. has following bridges :

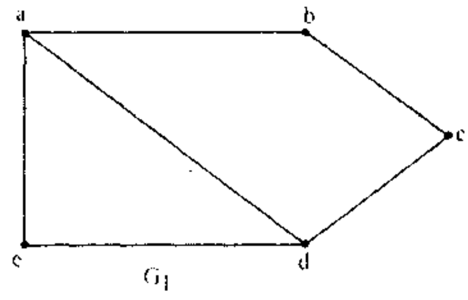
$\{e_3, e_1\}$

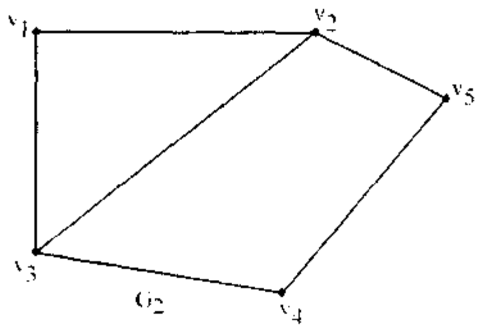
Q. 9. (b) Show that the following graphs are isomorphic.





Ans. Given graphs are :





If G_1 and G_2 are isomorphic graphs, then G_1 and G_2 have :

(i) same no. of vertices

(ii) same no. of edges

(iii) same degree sequences.

(i) No. of vertices in $G_1 = 5 =$ No. of vertices in G_2

(ii) No. of edges in $G_1 = 6 =$ No. of edges in G_2

(iii) Degree of each vertex in G_1 :

$d(a) = 3$	$d(e) = 2$
$d(b) = 2$	$d(d) = 3$
$d(c) = 2$	

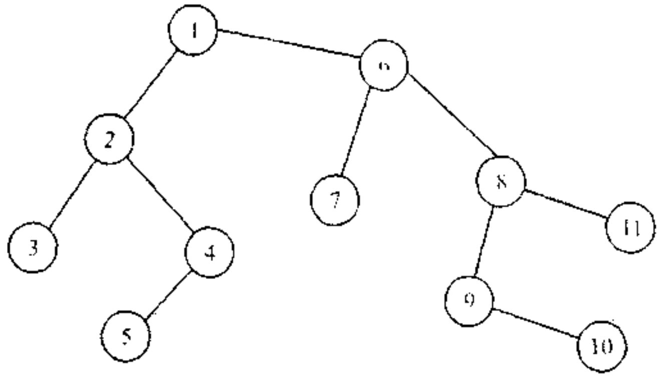
Degree of each vertex in G_2 :

$d(v_1) = 2$	$d(v_4) = 2$
$d(v_2) = 3$	$d(v_5) = 2$
$d(v_3) = 3$	

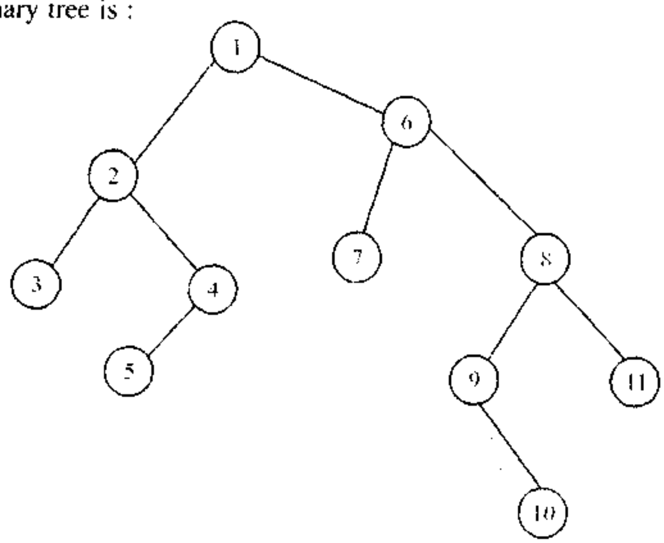
Since, G_1 and G_2 both have 3 vertices of degree 2 and 2 vertices of degree 3.

Therefore, G_1 and G_2 are isomorphic.

Q. 9. (c) Determine the preorder, postorder, and inorder traversal of the binary tree.



Ans. Given binary tree is :



Preorder Traversal : 1-2-3-4-5-6-7-8-9-10-11

Postorder Traversal : 3-5-4-2-7-9-10-11-8-6-1

Inorder Traversal : 3-2-5-4-1-7-6-9-10-8-11

Q. 9. (d) Draw the unique binary tree for the given inorder and postorder traversal.

Inorder :	**4**	**6**	**10**	**12**	**8**	**2**	**1**	**5**	**7**	**11**	**13**	**9**	**3**
Postorder :	**12**	**10**	**8**	**6**	**4**	**2**	**13**	**11**	**9**	**7**	**5**	**3**	**1**

Ans. Given :

Inorder :	4	6	10	12	8	2	1	5	7	11	13	9	3
Postorder :	12	10	8	6	4	2	13	11	9	7	5	3	1