B.E.

Third Semester Examination, May-2008

DISCRETE STRUCTURE

Note : Attempt any five questions.

Q. 1. (a) Let $S = \{1, 2, 3, 4\}$ and $A = S \times S$. Define the relation R on A as follows : we say that (a, b) R(a', b') iff $ab' = a'b$. Show that R is an equivalent relation. Compute A/R also.

Ans. $S = \{1,2,3,4\}$

$A = S*S$

$\Rightarrow \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$

{1, 1}, {1, 2}, {1, 3}, {1, 4}, {2, 1}, {2, 2}, {2, 3}, {2, 4}, {3, 1}, {3, 2}, {3, 3}, {3, 4}, {4, 1}, {4, 2}, {4, 2}, {4, 3}, {4, 4}.

Relation R is reflexive as {1, 1}, {2, 2}, {3, 3}, {4, 4} $\in R$.

Relation R is transitive as {1, 2} $\in R$ and {2, 1} $\Rightarrow \{1,1\} \in R$.

Symmetric : Relation R is symmetric because

$\{2,4\} \in R \Rightarrow \{4,2\} \in R$.

As R is reflexive, symmetric and transitive. Hence relation R is equivalence relations.

Q. 1. (b) Let $f: A \to B$ be a function. Then show that f^{-1} exists iff f is a bijective function.

Ans. A function $A \to B$ is invertible iff it is bijective function.

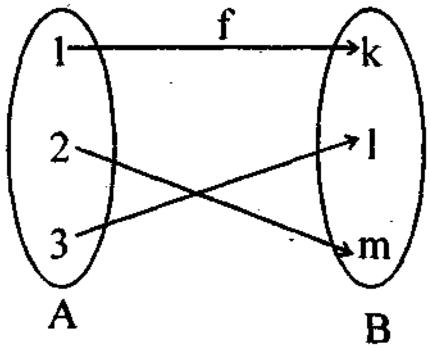
Consider the bijective function $f: A \to B$. As f is one to one, therefore each element of A corresponds to a distinct elements of B. As f is onto, therefore no element of B which is not the image of any element of A i.e. range = co-domain B.

The inverse function for f exists if f^{-1} is a function from $B \to B$.

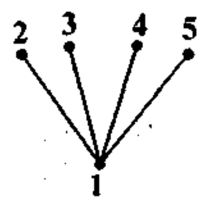
e.g., Let $A = \{1,2,3\}$,

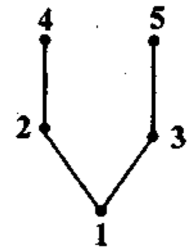
$B = \{k, l, m\}$ and $f: A \to B$ such that,

$f = \{(1,k),(2,m),(3,l)\}$

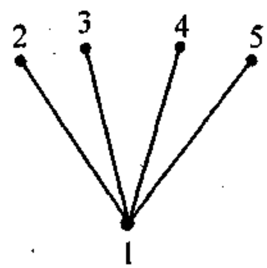


Q. 2. (a) Determine the matrix of the partial order whose Hass diagram are given as :

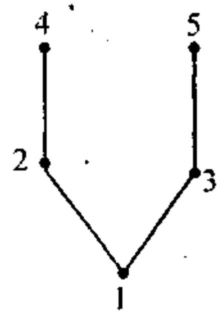




Ans.



	1	2	3	4	5
1	0	1	1	1	1
2	1	0	0	0	0
3	1	0	0	0	0
4	1	0	0	0	0
5	1	0	0	0	0



$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 0 & 1 \\ 4 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 \end{array}$$

Q. (b) Let $A = \{1, 2, 4, 8\}$ and let $\leq$ be the partial order of divisibility on A. Let A' = (0, 1, 2, 3) and let $\leq'$ be the usual relation "less than or equal to" on integers. Show that $(A, \leq)$ and $(A', \leq')$ are isomorphic posets.

Q. 2. (c) Show that if R is a partial order on the set A then R^{-1} is also a partial order on A. In particular, if R is a linear order then prove that R^{-1} is also a linear order.

Q. (a) Consider the following conditional statement :

p : If the flood destroys my house or the fire destroys my house, then my insurance company will pay me.

Find the converse and contrapositive of the statement p.

Ans. a : If load destroys my house

b : Fire destroys my house.

c : My insurance company will pay me.

The expression is,

$$a \vee b \rightarrow c$$

Contrapositive

$$\sim c \rightarrow \sim a \wedge \sim b$$

Converse

$$c \rightarrow a \vee v.$$

Q. 3. (b) Prove that each of the following is a tautology :

(i) $\big((p \Rightarrow q)\wedge(q \Rightarrow r)\big) \Rightarrow (p \Rightarrow r)$

(ii) $\sim(p \Rightarrow q) \Rightarrow p$

Ans. (i) $\big((p \Rightarrow q)\wedge(q \Rightarrow r)\big) \Rightarrow (p \Rightarrow r)$

p	q	$p \rightarrow q$	q	r	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$
T	T	T	T	T	T	T	T
T	F	F	T	F	F	F	F
F	T	T	F	T	T	T	T
F	F	T	F	F	T	T	F

Yes, the above statement is tautology.

(ii) $\sim(p \Rightarrow q) \Rightarrow p$

p	q	$p \rightarrow q$	$\sim(p \rightarrow q)$	p
T	T	T	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

The above statement is not tautology.

Q. 3. (c) Define : (i) Proposition, (ii) Quantifier (iii) Predicate (iv) Conjunction, (v) Disjunction.

Ans. (i) Proposition :

A proposition is a declarative statement that is either TRUE or FALSE, but not both.

- Propositions are T (true) or F (false).

- Corresponds to 1 and 0 in digital circuits.

- Proportions are typically represented by the letters : P, Q, R, S

- Logical Operators (Connectives)

- Negation (NOT)

- Conjunction (AND)

- Disjunction (OR)

- Exclusive or (XOR)

- Implication (IF-THEN)

- Biconditional (IF AND ONLY IF)

- Truth tables are used evaluate compound propositions made by using operators to combine propositions.

(ii) Quantifier :

(iii) Predicate : A predicate is a Unary Function whose result represents the truth the truth or falsehood of some condition. A predicate might, for example, be a function that takes an argument of type int and returns true if the argument is positive.

(iv) Conjunction : Conjunction (AND)

Binary Operator, Symbol : ^

- Def : Let P and Q be propositions. The conjunction of P and Q is denoted by $P \wedge Q$.

- The proposition is read "P or Q".

- If either P is TRUE or Q is TRUE then $P \vee Q$ is TRUE, otherwise the proposition is FALSE.

Q. 4. (a) Find an explicit formula for the sequence defined by $C_n = 3C_{n-1} - 2C_{n-2}$, $C_1 = 5$ and v.

Ans.

$$C_n = 3C_{n-1} - 2C_{n-2}$$

$$C_1 = 5,\ C_2 = 3$$

$$C_3 = 3C_2 - 2C_1$$

$$= 3*3 - 2*5 = -1$$

$$C_4 = 3C_3 - 2C_2$$

$$= 3*(-1) - 2*3 = -3 - 6 = -9$$

The sequence generated is $2(C_n - C_{n-1})$

Q. 4. (b) Determine the numeric function corresponding to each of the following generating functions :

(i) $G(x) = \frac{1}{5 - 6x + x^2}$,

(ii) $G(x) = \frac{2}{1 - 4x^2}$.

Ans.

Q. 4. (c) Find the particular solution of the difference equation $a_r + a_{r-1} = 5r2^r$.

Ans. $a_r + a_{r-1} = 5r2^r$

The above equation can be written as

$$(E^2 + E)a_r = 5r.2^r$$

The particular solution is given by

$$a_r = 5r.2^n.\frac{1}{(E^2 + E)}$$

$$= 2^r.5r\frac{1}{(2E)^2 + 2E}$$

$$= 2^r\left(4(1+\Delta)^2 + 2(1+\Delta)\right)^{-1}.5r$$

$$= 2^r\left(6 + 10\Delta + 4\Delta^2\right).5r$$

$$= \frac{2^r}{6}\left(1 + \frac{10\Delta + 4\Delta^2}{6}\right).5[r]$$

$$= \frac{2^r}{6}\left[1 + \frac{5\Delta + 2\Delta^2}{3}\right].5[r]$$

$$= 5.\frac{2^r}{6}\left[\{r\} - \frac{5}{3}\right]$$

$$a_r = \frac{5.2^n}{6}\left[r - \frac{5}{3}\right].$$

Q. 5. (a) Let G be the set of all non-zero real numbers. Define a binary operation * on G as : For $a, b \in G$, $a * b = \frac{ab}{2}$. Show that (G, *) is an abelian group.

Ans. Closure property :

The set G is closed under the operation * since, $a * b = \frac{ab}{2}$ is a real number. Hence, belongs to G.

Associative property :

The operation * is associative. Let a, b, $c \in G$ then we have

$$(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{(ab)c}{16} = \frac{a(bc)}{16} = \frac{abc}{16}$$

Identity :

To find the identity element, let us assume that e is a +ve real number. Then $e * a = a$ where $a \in G$.

$$\frac{ea}{4} = a \text{ or } e = 4.$$

Thus, the identity element in G is 4.

Inverse : Let us assume that $a \in G$. If $a^{-1} \in Q$ is an inverse of a, then $a * a^{-1} = 4$

$$\therefore \quad \frac{aa^{-1}}{4} = 4 \text{ or } a^{-1} = \frac{16}{a}.$$

Similarly, $a^{-1} * a = 4$

$$\therefore \quad \frac{a^{-1}a}{4} = 4 \text{ or } a^{-1} = 16/a$$

Thus, the inverse of element a in G is 16/a.

Commutative :

The operation * on G is commutative.

Since $$a * b = \frac{ab}{4} = b * a.$$

Thus, the algebraic system (G, *) is closed, associative, identity element, inverse and commutative. Hence, the system (G, *) is an abelian group.

Q. 5. (b) Define each of the following by giving one example in each case :

(i) Cyclic group,

(ii) Normal subgroup,

(iii) Coset,

(iv) Monoid,

(v) Field.

Ans. (i) Cyclic group :

In group theory, a cyclic group or monogenous group is a group that can be generated by a single element, in the sense that the group has an element g (called a "generator" of the group) such that, when written multiplicatively, every element of the group is a power of g (a multiple of g when the notation is additive).

(ii) Normal subgroup :

Normal subgroups are important because they can be used to construct quotient groups from a given group.

$$N \to G \Leftrightarrow \forall n \in N,\ g \in G\ gng^{-1} \in N$$

The following conditions are equivalent to requiring that a subgroup N be normal in G. Any one of them may be taken as the definition :

1. For all in G, gNg^{-1} N.

2. For all g in G, $gNg^{-1} = N$.

3. The sets of left and right cosets of N in G coincide.

4. For all g in G, $gN = Ng$.

5. N is a union of conjugacy classes of G.

6. There is some homomorphism on G for which N is the kernel.

Note that condition (1) is logically weaker than condition (2) and condition () is logically weaker than condition (4). For this reason, conditions (1) and (3) are often used to prove that N is normal in G, while conditions (2) and (4) are used to prove consequences of the normality of N in G.

(iii) Coset : In mathematics, if G is a group, H a subgroup of G and g and element of G, then

$gH = \{gh : h \text{ an element of } H\}$ is a left coset of H in G, and

$Hg = \{hg : h \text{ an element of } H\}$ is a right coset of H in G.

Only when H is normal will the right and left cosets of H coincide, which is one definition of normality of a subgroup.

A coset is a left or right coset of some subgroup in G. Since $Hg = g(g^{-1}Hg)$, the right cosets Hg (of H) and the left cosets $g(g^{-1}Hg)$ (of the conjugate subgroup $g^{-1}Hg$) are the same. Hence it is not meaningful to speak of a coset as being left or right unless one first specifies the underlying subgroup.

For abelian groups or groups written additively, the notation used changes to g+H and H+g respectively.

(iv) Monoid :

A monoid is a set that is closed under an associative binary operation and has an identity element $I \in S$ such that for all $a \in S$, $Ia = aI = a$. Note that unlike a group, its elements need not have inverses. It can also be thought of as a semigroup with an identity element.

A monoid must contain at least one element.

A monoid that is commutative is, not surprisingly, known as a commutative monoid.

(v) Field :

In abstract algebra, a field is an algebraic structure in which the operations of addition, subtraction, multiplication and division (except division by zero) may be performed, and the same rules hold which are familiar from the arithmetic or ordinary numbers.

All fields are rings, but not conversely. Fields differ from rings most importantly in the requirement that division be possible, but also, in modern definitions, by the requirement that the multiplication operation in a field be commutative. Otherwise the structure is a so-called skew field (better known as a division ring), although historically division rings were called fields and fields were commutative fields.

The prototypical example of a field is Q, the of rational numbers. Other important examples include the field of real numbers R, the field of complex numbers C and for any prime number p, the finite field of integers modulo p, denoted Z/pZ., F_p or $GF(p)$. For any field K, the set K(X) of rational functions with coefficients in K is also a field.

The mathematical discipline concerned with the study of fields is called field theory.

A field is a specific type of integral domain, and can be characterized by the following (not necessarily exhaustive) chain of class inclusions.

Q. 6. (a) How many integers are there between 5 and 1004 that are multiple of 3?

Ans.

Q. 6. (b) Find the number of distinguishable words that can be formed from the letters of MISSISSIPPI.

Ans. There are 11 letters in the word.

$\therefore$ The total number of permutations of these letters 11!.

$$= 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= 39916800$$

Q. 6. (c) A box contains six white balls and five red balls. In how many ways, four balls can be drawn from the box if ;

(i) They can be of any colour,

(ii) Two balls are white and two red,

(iii) All the balls are of same colour.

Ans. White ball = 6

Red balls = 5

(i) There are 11 balls and 4 are to be selected

$$= {}^{11}P_4 = \frac{11!}{(11-4)!} = \frac{11!}{7!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7!}{7!} = 7920.$$

(ii) There are 6 white balls and 5 red balls.

$${}^{6}P_2 + {}^{5}P_2$$

$$= \frac{6!}{(6-2)!} + \frac{5!}{(5-2)!} = \frac{6!}{4!} + \frac{5!}{3!}$$

$$= \frac{6\times5\times4!}{4!} + \frac{5\times4*3!}{3!}$$

$$= 30+20 = 50$$

(iii) All balls of same colours.

$${}^6P_4 + {}^5P_4$$

$$= \frac{6!}{(6-4)!} + \frac{5!}{(5-4)!} = \frac{6!}{2!} + \frac{5!}{0}$$

$$= \frac{6\times5\times4\times3*2!}{2!} + 5!$$

$$= 360+120 = 480$$

Q. 7. (a) State Koingsberg seven bridger problem. What is the solution of this problem? Elaborate.

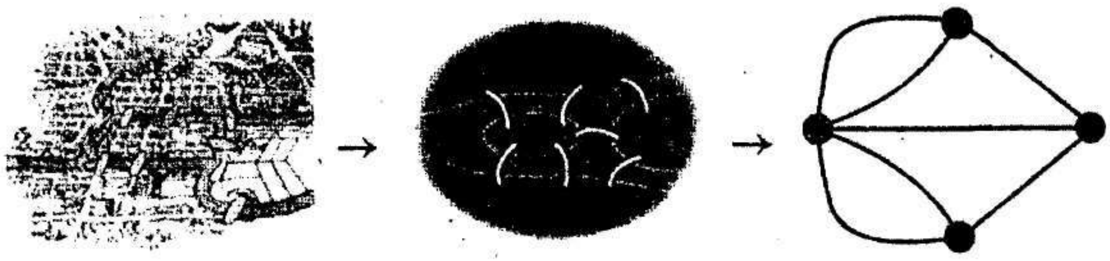
Ans. Koingsberg seven bridger problem :

Leonhard Euler proved that it was not possible. In proving the result, Euler formulated the problem in terms of graph theory, by abstracting the case of Konigsberg—first, by eliminating all features except the landmasses and the bridges connecting them; second, by replacing each landmass with a dot, called a vertex or node, and each bridge with a line, called an edge or link. The resulting mathematical structure is called a graph.

The shape of a graph may be distorted in any way without changing the graph itself, so long as the links between nodes are unchanged. It does not matter whether the links are straight or curved, or whether one node is to the left or right of another.

Euler realized that the problem could be solved in terms of the degrees of the nodes. The degree of a node is the number of edges touching it; in the Konigsberg bridge graph, three nodes have degree 3 and one has degree 5. Euler proved that a circuit of the desired form is possible if and only if the graph is connected, and there are exactly two or zero nodes of odd degree. Such a walk is called an Eulerian path or Euler walk. Further, if there are two nodes of odd degree, those must be the starting and ending points of an Eulerian path. Since the graph corresponding to Konigsberg has four nodes of odd degrees, it cannot have an Eulerian path.

An alternative form of the problem asks for a path that traverses all bridges and also has the same starting and ending point. Such a walk is called an Eulerian circuit or an Euler tour. An Eulerian circuit exists if and only if the graph is connected and there are no nodes of odd degrees. It can be seen that all Eulerian circuits are also Eulerain paths.



Q. 7. (b) Evaluate the following expression, given in prefix notation :

(i) $\times - +34 - 72 \div 12 \times 3 - 64$

(ii) $\times - 64 + 5 \div 22$.

Ans. (i) $\times - +34 - 72 \div 12 \times 3 - 64$

$$* - +34 - 72 \div 12 * 3 - 64$$

$$= * - (+34)(-72) \div 12 * 3(-64)$$

$$= *(-38) \div 12(-192)$$

$$= *(-38) - (-16)$$

$$= 608 .$$

(ii) $\times - 64 + 5 \div 22$:

$$* - 64 + 5 \div 22$$

$$= *(-64)(+5) \div 22$$

This is incorrect expression.

Q. 7. (c) Define by giving one example each :

(i) Cut points,

(ii) Bridge,

(iii) Multigraph

(iv) Spanning tree.

Ans. (i) Cut points :

A cut-point is a point of a connected space such that its removal causes the resulting space to be disconnected. For example every point of a line is a cut-point, while no point of a circle is a cut-point. Cut-points are useful in the characterization of topological continua, a class of spaces which combine the properties of compactness and connectedness and include many familiar space such as the unit interval, the circle, and the torus.

(ii) Bridge :

(iii) Multigraph :

A multigraph or pseudograph is a graph which is permitted to have multiple edges, (also called "parallel edges"). that is, edges that have the same end nodes. Thus, two vertices may be connected by more than one edge. Formally, a multigraph G is an ordered pair G : =(V, E) with

- V a set of Vertices or nodes,

- E a multiset of unordered Paris of vertices, called edges or lines.

Multigraphs might be used to model the possible flight connections offered by an airline. In this case the pseudograph would be a directed graph with pairs of directed parallel edges connecting cities to show that it is possible to fly both to and from these locations.

Some authors also allow multigraphs to have loops, that is, an edge that connects a vertex to itself.

A multigraph is a directed graph which is permitted to have multiple arcs, i.e., arcs with the same source and target nodes. A multidigraph G is an ordered pair G : = (V, A) with

- V a set of vertices or nodes,

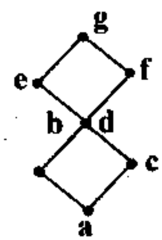
- A a multiset of ordered pairs of vertices called directed edges, arcs or arrows.

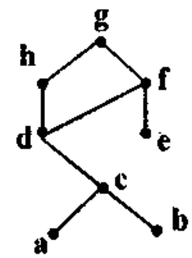
In category theory a small category can be defined as a multidigraph equipped with an associative composition law and a distinguished self-loop at each vertex serving as the left and right identity for composition. "Graph" is standardly taken to mean "multigraph" in category theory for this reason, and the underlying graph of a category refers to its underlying multidigraph. A mixed multigraph G : = (V, E, A) may be defined in the same way as a mixed graph.

(iv) Spanning tree :

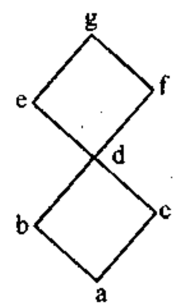
Spanning tree T of a connected, undirected graph G is a tree composed fo all the vertices and some (or perhaps all) of the edges of G. Informally, a spanning tree of G is a selection of edges of G that form a tree spanning every vertex. That is, every vertex lies in the tree, but no cycles (or loops) are formed. In the other hand, every bridges of G must belong to T. A spanning tree of a connected graph G can also be defined as a maximal set of edges of G that contains no cycle, or as a minimal set of edges that connect all vertices. In certain fields of graph theory it is often useful to find a minimum spanning tree of a weighted graph. Other optimization problems on spanning trees have also been studied, including the maximum spanning tee, the minimum tree that spans at least k vertices, the minimum spanning tree with at most k edges per vertex (MDST), the spanning tree with the largest number of leaves (closely related to the smallest connected dominating set), the spanning tree with the fewest leaves (closely related to the smallest connected dominating set), the spanning tree with the fewest leaves (closely related to the Hamiltonian path problem), the minimum diameter spanning tree, and the minimum dilation spanning tree.

Q. 8. (a) Determine whether the following Hasse diagrams represent lattices, (give reason).



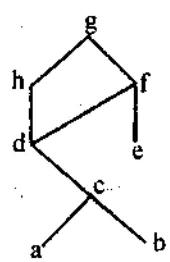


Ans. (i)



The above Hasse diagram is representing lattices.

(ii)

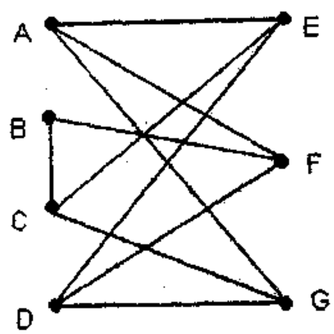


The above Hasse diagram is representing lattices.

Q. 8. (b) State Euler formula for connected planar graphs and illustrate it for two such graphs.

Ans. Euler formula : In graph theory, a planar graph is a graph which can be emedded in the plane, i.e., it can be drawn on the plane in such a way that its edges may intersect only at their endpoints. A nonplanar graph is the one which cannot be drawn in the plane without edge intersections. A planar graph already drawn in the plane without edge intersections is called a plane graph or planar embedding of the graph. A plane graph can be defined as a planar graph with a mapping from every node to a point in 2D space and from every edge to a plane curve, such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points. It is easily sent hat a graph that can be drawn on the plane can be drawn on the sphere as well, and vice-versa. The equivalence class of topologically equivalent drawings on the sphere is called a planar map. Although a plane graph has an external or unbounded face, none of the faces of a planar map have a particular status.

A generalization of planar graphs are graphs which can be drawn on a surface of a given genus. In this terminology, planar graphs have graph genus 0, since the plane (and the sphere) are surfaces of genus 0.



Euler's formula states that if a finite, connected, planar graph is drawn in the plane without any edge intersections and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely-large region), then

$$v - e + f = 2$$

i.e. the Euler characteristic is 2. As an illustration, in the first planar graph given above, we have v = 6, e = 7 and f = 3. If the second graph is redrawn without edge intersections, we get v = 4, e = 6 and f = 4. Euler's formula can be proven as follows : If the graph isn't a tree, then remove an edge which completes a cycle. This lowers both e and f by one, leaving $v - e + f$ constant. Repeat until you arrive at a tree; trees have $v = e + 1$ and f = 1, yielding $v - e + f = 2$.

In a finite, connected, simple, planar graph, any face (except possibly the outer one) is bounded by at least three edges and every edge touches at most two faces; using Euler's formula; one can then show that these graphs are sparse in the sense that $e \le 3v - 6$ if $v \ge 3$.

A simple graph is called maximal planar if it is planar but adding any edge would destroy that property. All faces (even the outer one) are then bounded by three edges, explaining the alternative term triangular for these graphs. If a triangular graph has v vertices with $v > 2$, then it has precisely 3v-6 edges and 2v-4 faces.

Euler's formula is also valid for simple polyhedra. This is no coincidence : every simple polyhedron can be turned into a connected, simple, planar graph by using the polyhedron's vertices as vertices of the graph and the polyhedron's edges as edges of the graph. The faces of the resulting planar graph then correspond to the faces of the polyhedron. For example, the second planar graph shown above corresponds to a tetrahedron. Net every connected, simple, planar graph belongs to a simple polyhedron in this fashion; the trees do not, for example. A theorem of Earnest Steinitz says that the planar graphs formed from convex polyhedra (equivalently : those formed from simple polyhedra) are precisely the finite 3-connected simple planar graphs.